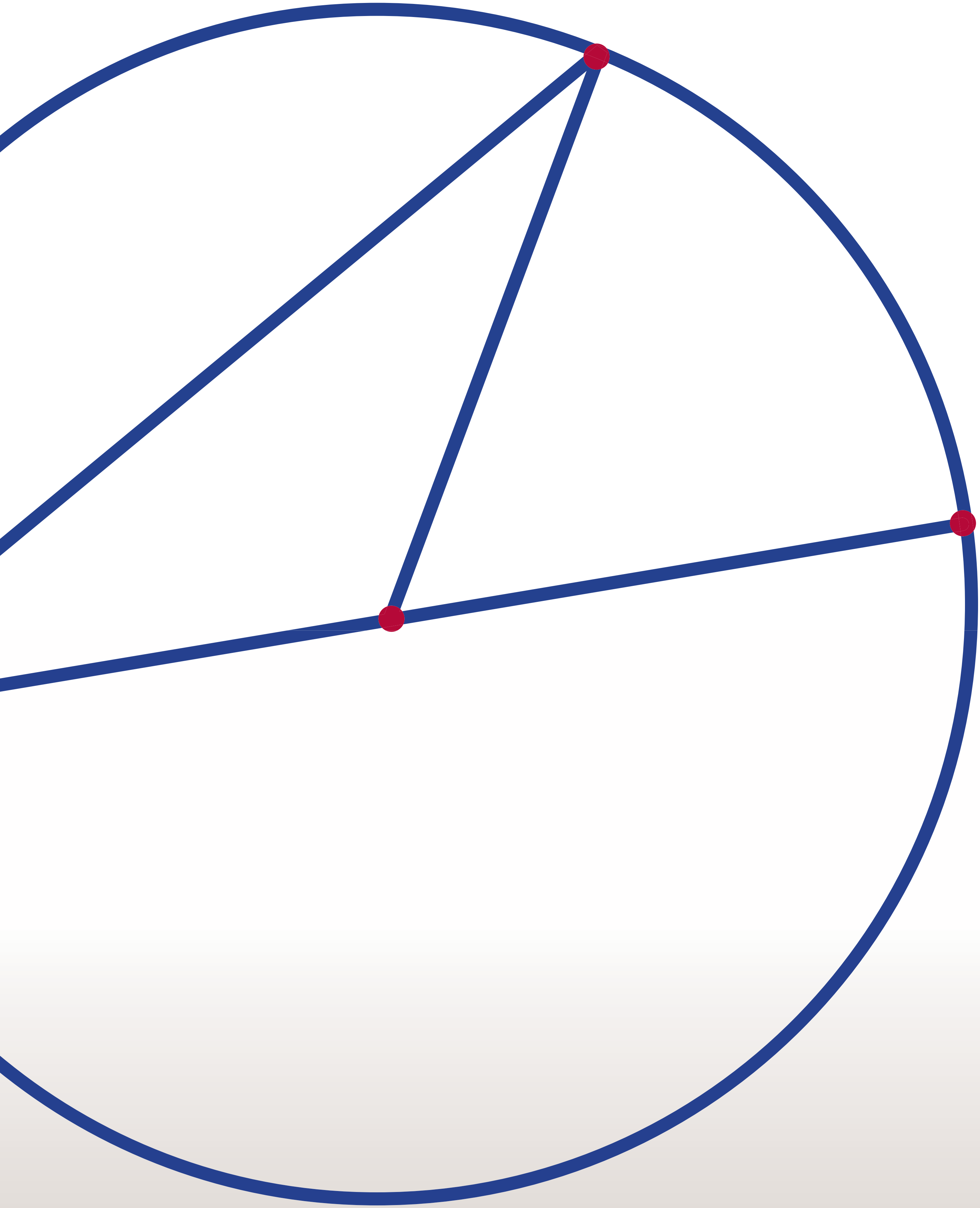
IMPROVING STUDENT REASONING IN GEOMETRY

Parallel geometry tasks with four levels of complexity involve students in writing and understanding proof.

Bobson Wong and Larisa Bukalov

In our years of teaching geometry, the greatest challenge has been getting students to improve their reasoning. Many students have difficulty writing formal proofs—a task that requires a good deal of reasoning. Proof is a problem-solving activity, not a procedure that can be done routinely (Cirillo 2009). We wanted to find ways to scaffold our instruction to prepare students for harder problems.

In planning geometry lessons, we noticed that many problems that we selected could be arranged from very straightforward (often a simple diagram with a missing quantity) to very complex (such as a detailed formal proof). Tiering the lessons—that is, creating multiple pathways for students to understand the goals of a lesson—might be the best strategy (Pierce and Adams 2005). The work of van Hiele (1986) and others building on van Hiele's work, such as Burger and Shaughnessy (1986) and Clements (2003), was extremely helpful in suggesting that geometry requires higher-order thinking and that students need more experience with lower levels of thinking before they can succeed at higher levels. Following the recommendation of Artzt et al. (2008), we wanted to present problems in a way that was accessible enough for students to use their prior knowledge but also challenging enough so that they could extend their learning.



Although sequencing problems properly is important, we knew that sequencing alone would not guarantee that all students would be able to improve their reasoning. We wanted all students to have an equal opportunity to improve, and we wanted to avoid the dangers of "tracking" students by ability. In tracked classes, lower-level students often have limited exposure to a high-quality mathematics education (Useem 1990). We wanted a system that was fair and flexible.

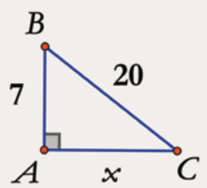
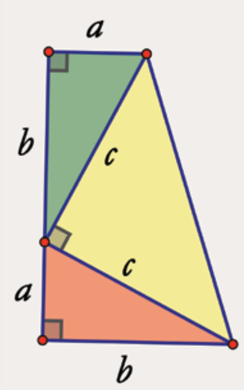
Our solution was to divide the lessons into parallel tasks, allowing students with different levels of understanding of a topic to work on the same task simultaneously (Small and Lin 2010). We organized problems into four levels of complexity but allowed students to select their own level and move freely between levels. By having access to all levels, students could monitor their own progress and know what they needed to do to move to the next level. We fit our lessons within a familiar three-part framework: whole-group introductory discussion, guided independent practice, and whole-group summary. At the same time, our model was simple enough for teachers, students, and parents to understand.

THE FOUR LEVELS OF COMPLEXITY

Our organizational model divides problems into four levels with questions of increasing complexity. These four levels were inspired by the four-point system used to report our statewide exam questions as well as the four Depth of Knowledge levels (Webb, Vesperman, and Ely 2005). **Table 1** contains a brief, general description of levels of problems in our model, along with a sample problem for each level for a lesson on the Pythagorean theorem.

Level 1, the simplest level, consists of problems that can be solved by directly applying a fact, method, or formula. Problems at this level typically do not require students to use precise mathematical language. For example, the level 1 problem in **table 1** is a straightforward application of the Pythagorean theorem. From the diagram, students can immediately identify the two legs and the hypotenuse of the triangle and apply the formula $a^2 + b^2 = c^2$ without having to name the theorem or explain why triangle *ABC* is a right triangle. Other examples of level 1 problems include finding the midpoint of a line segment given the coordinates of its endpoints or stating the properties of a parallelogram.

Table 1 Levels of Complexity for Problems

Level	Characteristics	Sample Problem for Lesson on Pythagorean Theorem
1	Student solves problem by directly recalling a fact, method, or formula.	Find the value of x in simplest radical form. 
2	Student solves problem through an additional step beyond a level 1 problem. Typically, some guidance about the method needed for solving is provided.	The lengths of three sides of a triangle are 25, 7, and 24. Determine whether the triangle is a right triangle.
3	Student solves problem by selecting the appropriate pieces of information independently (usually two or more definitions, theorems, formulas, or methods).	In an isosceles trapezoid, the lengths of the bases are 14 in. and 30 in. The length of each of the nonparallel sides is 10 in. Find the length of the altitude of the trapezoid.
4	Student solves problem by using deductive reasoning to prove mathematical statements.	Explain how the diagram at right can be used to prove the Pythagorean theorem. 

Level 2 problems require an additional step beyond a level 1 problem. For example, level 2 problems may require students to draw an accurately labeled diagram when none is provided. A level 2 problem may also require translating a simple word problem into an algebraic representation. To solve the level 2 problem shown in **table 1**, students must recognize that if the given triangle is right, then the side length of 25 must be the hypotenuse, the longest side of a right triangle. Level 2 problems typically provide some instruction about the method required to solve the problem, such as "Determine whether the triangle is a right triangle" or "Use the midpoint formula to determine whether the quadrilateral is a parallelogram."

Level 3 problems require students to determine independently what information is needed for a solution—typically, several formulas, theorems, or facts. The level 3 question in **table 1** requires students to integrate several ideas: drawing an appropriately labeled diagram, recognizing that the altitudes divide the trapezoid into right triangles and a rectangle, and then applying the Pythagorean theorem and properties of rectangles to find the length of the altitude. Level 3 problems can also include interpreting complex multistep diagrams, a task that requires the application of several theorems. Problems at this level do not require applying deductive reasoning to write formal Euclidean proofs. However, level 3 problems can ask students to select appropriate solution methods and justify their calculations by citing appropriate definitions or theorems. For example, a level 3 problem could give the coordinates of a quadrilateral and ask students to prove that it is a parallelogram.

Level 4 problems require students to use deductive reasoning to prove mathematical statements. Students must have reasoning skills that are strong enough for writing formal proofs. Problems at this level include formal Euclidean proofs.

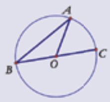
SAMPLE LESSON WITH PARALLEL TASKS: INSCRIBED ANGLES

By dividing classwork into four levels, we provide multiple entry points for students. Problems from all four levels were included on one activity sheet distributed to all students so that they could see questions from all levels. However, students would have to master one level before moving on to the next. Following Small and Lin's (2010) advice, we allow students to select the level that they feel is most appropriate for their readiness for the lesson. We believe that allowing students to select the appropriate starting point for their work empowers them. Many weaker students told us that they did not feel stigmatized by starting at lower levels because they were able to get more practice to work

AIM # ________ : What are the properties of inscribed angles? **DATE: ________________**

Show all work on a separate piece of paper. Attach this sheet to the front of your work.

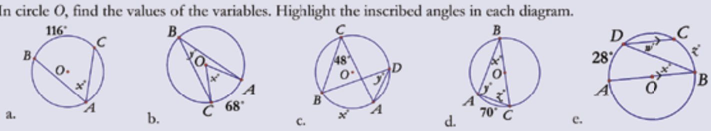
DO NOW



1. Circle O has diameter $\overline{BOC}$ and radius $\overline{OA}$. If $m\angle ABO = 30$, find the following angle measures (and state the definition or theorem that justifies each answer):
   a. $m\angle BAO$ (HINT: What kind of triangle is $\triangle BAO$?) b. $m\angle AOC$ c. $m\overset{\frown}{CA}$
2. What is the relationship between $m\angle AOC$ and $m\angle ABC$? (HINT: What is the ratio of the two angle measures?)
3. If $m\angle ABO = 40$, would the relationship between $m\angle AOC$ and $m\angle ABC$ change? Explain.
4. Fill in the blanks: The Do Now illustrates the following:
   INSCRIBED ANGLE THEOREM: The measure of an inscribed angle is ________ the measure of its ____________ arc.

LEVEL 1

5. In circle O, find the values of the variables. Highlight the inscribed angles in each diagram.



6. Problems 5c, 5d, and 5e above illustrate the following theorems. Fill in the blanks in each sentence.
   a. If two inscribed angles of a circle intercept the same arc, then the angles are ______________ .
   b. An angle inscribed in a semicircle is a ____________ angle.
   c. If _______ lines intersect a circle, then the arcs on the circle cut off by the lines are __________ .

LEVEL 2

7. An inscribed angle and a central angle intercept the same arc on circle O. Find the ratio of the measure of the central angle to the measure of the inscribed angle.
8. The vertices of an inscribed triangle divide the circle into three arcs whose measures are in the ratio 5 : 6 : 7. Find the measure of the smallest angle of the triangle.
9. Points P, Q, and R lie on circle O, and $m\angle POQ = 68$. Find $m\angle PRQ$.
10. Quadrilateral $ABCD$ is inscribed in circle O (see figure). Find the values of w, x, y, and z.
    a. This problem illustrates the theorem (fill in the blanks): If a ____________ is inscribed in a _________ , then its opposite angles are ______________ .
11. Quadrilateral $EFGH$ is inscribed in circle O and $m\angle EFG = 70$. Find $m\angle GHE$.

Ex. 10

LEVEL 3

12. In circle O, quadrilateral $ABCD$ is inscribed, $m\angle A = x^2 + 140$, and $m\angle C = 3x$. Find $m\angle A$, $m\angle B$, $m\angle C$, and $m\angle D$, if possible. Justify each answer or explain why the measure cannot be found.
13. In circle O, $\triangle ABC$ is inscribed; radii $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are drawn; and $m\angle ACO = 35$. Find $m\angle AOC$ and $m\angle ABC$. Justify your answer.
14. In circle O, diameters $\overline{AFOD}$ and $\overline{EOC}$ are drawn, $\triangle ADC$ is inscribed, chord $\overline{EFB}$ is drawn, B lies on circle O, $m\angle AFB = 100$, and $m\angle COD = 60$. Find $m\angle BEC$. Justify your answer.

LEVEL 4

15. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. Prove that $\triangle ADE \sim \triangle CBE$ and $CE \cdot DE = BE \cdot EA$.
16. Prove the theorems stated in problems 6 and 10b.
17. Prove the Inscribed Angle Theorem. (HINT: Consider three cases: the center of the circle is on one side of the inscribed angle, the center is in the interior of the angle, and the center is in the exterior of the angle.)

SUMMARY

18. Parallelogram $ABCD$ is inscribed in circle O.
    a. Draw a diagram illustrating this sentence. Be as accurate as possible.
    b. What do you notice about the parallelogram? (HINT: What kind of parallelogram must it be?)
    c. Justify the conjecture you made in part *b*. (HINT: Use at least one of the theorems you learned today.)

Fig. 1 This sample activity sheet includes questions from all four levels.

up to higher levels. Although many students who selected a level beyond their understanding for that topic soon chose a lower level, others found that they could handle a more challenging level than they originally thought. Whenever possible, we encourage students to start at a higher level if they find the lower-level questions too easy.

To give a better idea of what typical level 4 classwork looks like, we have included a sample activity sheet (see **fig. 1**). Samples of student work are also included to illustrate the level of detail and type of thinking expected at each level. This activity sheet provides not only practice problems but also enough guided questions (such as nos. 4, 6, and 10) so that students can learn new information without much direct instruction. Students who started at levels 3 or 4 were encouraged to review the lower-level problems to make sure that they knew how to do them and get relevant information (such as theorems and formulas) for higher-level problems.

To help students work independently and make the class flow more smoothly, we designed the

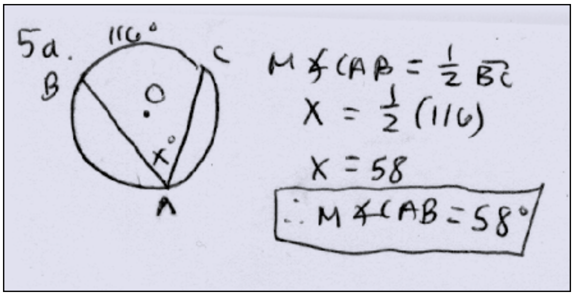


Fig. 2 A student response to problem 5(*a*) involves only applying the appropriate theorem and performing the calculation without justification.

tasks so that each student worked on the same topic simultaneously. This approach allowed all students to participate in a common discussion at the beginning and the end of class. We also posted hints and answers to problems both on the board and online so that students could check their work themselves in class or at home. In addition, we found that students working at a lower level sought help from others who had already completed those problems or were working on a higher level, thus freeing us to circulate around the room and help individual students more.

The Do Now portion of the activity sheet, which all students complete at the beginning of class, reviews relevant prior knowledge and introduces the new material in the lesson. In the example shown in **figure 1**, the Do Now task reviews previously learned theorems about the relationship between central angles and arcs and elicits the inscribed angle theorem from numerical examples and a fill-in-the-blank statement. This section is easy enough for students at all levels to complete and also provides enough additional information for them to start the new work.

The next problems are divided into four levels of complexity. In general, the lesson's most basic concepts are introduced in the Do Now assignment and level 1 questions; more advanced concepts are presented in levels 2 and 3; and the concepts required for full mastery are given in level 4. Each level contains practice questions appropriate for that level of difficulty. Each level can also contain enrichment questions or questions that introduce more difficult concepts for the next level.

Level 1 questions contain direct applications of the inscribed angle theorem, which was introduced in the Do Now task. **Figure 2** shows student work for problem 5(a)—a straightforward calculation applying the theorem with no explanation required. Following the advice of Cirillo (2009), we wanted to encourage students at all levels of understanding to make conjectures. Thus, this level contains a summary question (no. 6) that requires students to explain in words what they see in the examples

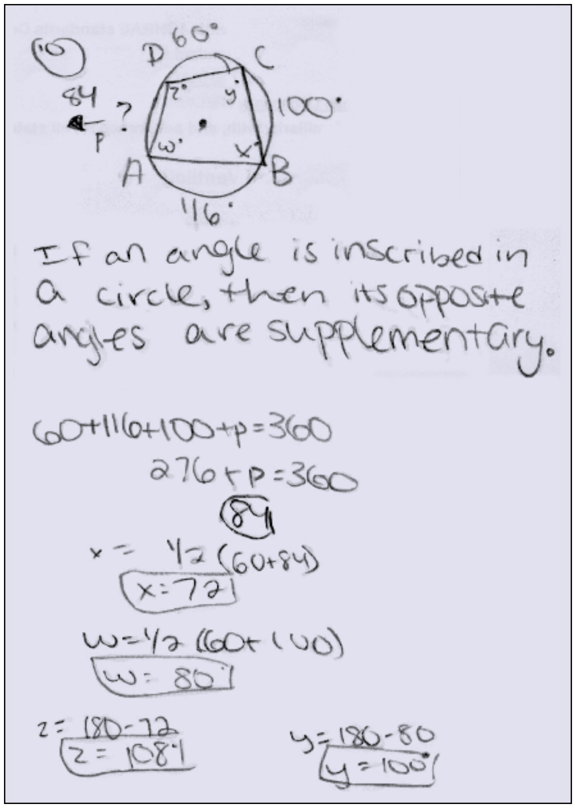


Fig. 3 This level 2 question requires students to draw the diagram and make conjectures. Note the student's vocabulary error.

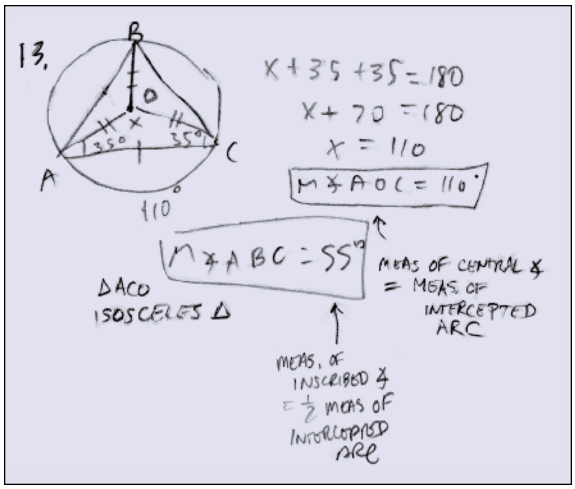


Fig. 4 Answering question 13 requires applying more than one theorem, a characteristic requirement of a level 3 question.

by filling in blanks in sentences. This question also introduces theorems used for other levels without requiring students to prove them.

Level 2 questions require the additional step of translating words into algebraic expressions or appropriately labeled diagrams. For example, question 9 is similar to 5(b), a level 1 problem, but lacks a diagram. Question 10 allows students to make further conjectures that lead to a theorem (see **fig. 3**).

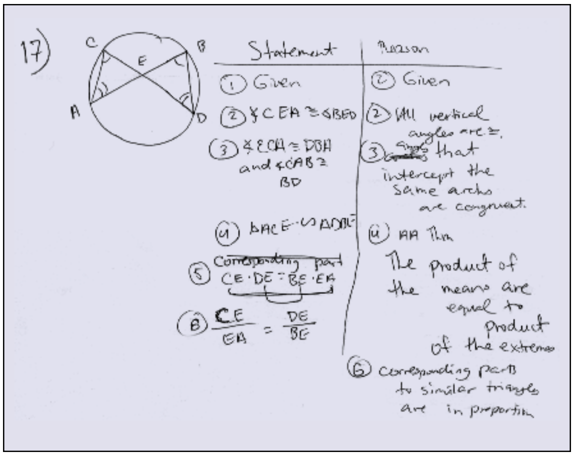


Fig. 5 Although this student has the right general idea, a letter is missing in statement 3, and the proportion is wrong in statement 6. Most teachers would insist that statement 6 precede statement 5.

Level 3 questions require students to use several theorems, formulas, or ideas in the same problem. For example, to answer question 13, students must apply the inscribed angle theorem as well as other theorems. Students can justify their answers to level 3 questions by annotating their work with appropriate definitions or theorems, as shown in the student work in **figure 4**.

Level 4 questions help students summarize the lesson by asking them to prove the theorems elicited in the previous levels. **Figure 5** shows an example of a student's formal proof for question 17.

The activity sheet concludes with a summary question (no. 18) that all students should be able to answer. In this sample, all students, no matter what level they complete, should be able to conjecture that a rectangle is the only parallelogram that can be inscribed in a circle. **Figure 6** shows student work that reflects different levels of complexity. Some students were able to justify their work only with a picture but lacked the precise mathematical language to write an explanation (see **figs. 6a** and **6b**). Other students were able to label diagrams with more information and write brief explanations (see **fig. 6c**), whereas some students were able to write a more formal proof. Because most students were able to illustrate the problem, we concluded the classes with a whole-group discussion in which all students could participate.

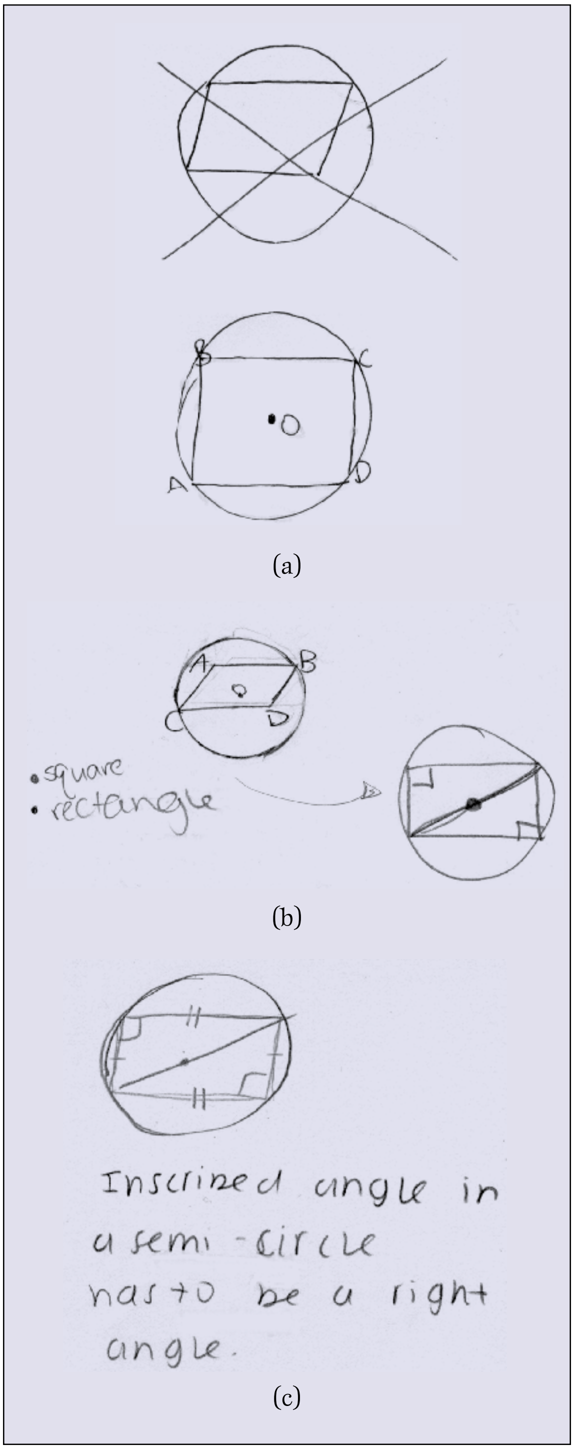


Fig. 6 These three examples of student work show differing levels of success.

CHALLENGES AND ADVANTAGES OF THE MODEL

While implementing this four-level model of parallel tasks in our classroom, we encountered several challenges. This model is not appropriate for every lesson. Some topics, such as introducing formal proofs, require a great deal of direct instruction that would be difficult to accomplish solely through parallel tasks. In addition, some students need help with selecting an appropriate level. We tested this model only for geometry, so we have not yet determined how it could be applied to other courses.

The greatest challenge with using this model is that planning effective lessons takes much more time and effort. We had to think much more carefully about what questions we asked. To determine each problem's level, we had to examine the type of work required in the solution. In addition, we had to balance the amount of work for each level so that all students, no matter where they began, would be sufficiently challenged in class. If some levels required much less time to finish than others, then some students would finish early, whereas others would be "stuck" at their level. Fortunately, we did not have to create all the problems from scratch. For many topics, we were able to use problems from the textbook; we simply organized these by level.

Another challenge is assessing student work. We assessed classwork informally to give students the freedom to answer questions from different levels. During class, we circulated around the room to monitor progress and help students when necessary. Simply checking the number of questions completed did not accurately tell us what students understood; they could have been confused or discouraged by one question. However, talking to each student individually helped us determine what problems the student had with the material. Over time, these informal conversations enabled us to see trends in each student's work but required a great deal of class time. Teachers with limited class time may need to devise other ways to assess student work.

Although this model requires a great deal of effort to implement, we believe that it is worth the investment. It has helped us communicate our expectations more clearly to both students and parents. Although we have not done a formal study of this model, informal conversations with students and parents indicate that they appreciate knowing what specific work needed to be done for improvement. We were able to give more structured and specific feedback to students and parents about what students knew about a particular topic. And by labeling each problem's level of difficulty, we helped avoid giving too much work that tests only low levels of understanding or too much work that lacks proper development. Organizing problems in this way was particularly helpful for student teachers, who often struggled with creating work that had an appropriate level of difficulty.

This model of parallel tasks can improve student reasoning because it clearly shows what is required to achieve mastery. By allowing students to choose appropriate levels of work every day, we empower them to take more control of their learning. Creating an effective lesson using parallel tasks takes a great deal of time and effort. However, using the parallel tasks model—even for only a few lessons—can be a valuable experience for both teachers and students.

BIBLIOGRAPHY

Artzt, Alice F., Eleanor Armour-Thomas, and Frances R. Curcio. 2008. *Becoming a Reflective Mathematics Teacher: A Guide for Observations and Self-Assessment*. 2nd ed. New York: Taylor and Francis.

Burger, William F., and J. Michael Shaughnessy. 1986. "Characterizing the van Hiele Levels of Development in Geometry." *Journal for Research in Mathematics Education* 17 (1): 31–48.

Cirillo, Michelle. 2009. "Ten Things to Consider When Teaching Proof." *Mathematics Teacher* 103 (4): 251–57.

Clements, Douglas H. 2003. "Teaching and Learning Geometry." In *A Research Companion to Principles and Standards for School Mathematics,* edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 151–78. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Pierce, Rebecca L., and Cheryll M. Adams. 2005. "Using Tiered Lessons in Mathematics." *Mathematics Teaching in the Middle School* 11 (3): 144–49.

Small, Marian, and Amy Lin. 2010. *More Good Questions: Great Ways to Differentiate Secondary Mathematics Instruction.* New York: Teachers College Press.

Useem, Elizabeth L. 1990. "You're Good, but You're Not Good Enough: Tracking Students out of Advanced Mathematics." *American Educator* 14 (3): 24–27, 43–46.

van Hiele, Pierre M. 1986. *Structure and Insight*. New York: Academic Press.

Webb, Norman L., Brian Vesperman, and Rob Ely. 2005. "Web Alignment Tool." Wisconsin Center of Educational Research, University of Wisconsin-Madison. http://wat.wceruw.org/index.aspx.

BOBSON WONG, bwong3@schools.nyc.gov, and **LARISA BUKALOV,** Lbukalo@schools.nyc.gov, teach mathematics at Bayside High School, a public high school in New York City. They are two-time recipients of the Math for America Master Teacher Fellowship.